7 Online Appendix C: Existence of Equilibria, Mixed Equilibria and Multiplicity of Equilibria

In this appendix, we prove the existence of an equilibrium in our main model, and show the possibility of open sets of parameters with mixing equilibria, asymmetric equilibria and multiple equilibria. Note that since the planner optimum generically involves full effort on a unique invention, the existence of these alternative equilibria do not in any way change our efficiency results. For simplicity, we show all examples using baseline firm transfers.

7.1 Equilibrium Existence

Consider first the problem of existence. Since the invention graph is finite, we can use best responses to compute equilibria by backward induction. Consider the stage game and take the continuation values $V_i(s)$ as given. To prove equilibrium existence, we use the following result: a symmetric game whose strategy set S is a nonempty, convex, and compact subset of some Euclidean space, and whose utility functions $u(s_i, s_1, \ldots, s_N)$, continuous in $(s_1, \ldots, s_N)$ and quasiconcave in s_i, has a symmetric pure-strategy equilibrium.[23]

Consider a formulation where the strategy space for each firm is the simplex $\Delta^{|S|}$. The firm's payoff, taking rival effort a_{-i} as given, can be simplified as

$$u(x_i, a_{-i}) = \frac{\sum_{s' \in S}[\alpha_{s'} x_{s'} + B(a_{-i,s'}, V_{is'})]}{\sum_{s' \in S}[\beta_{s'} x_{s'} + C(a_{-i,s'}, r)]}.$$

This is a linear fractional is own-strategy x, and linear fractionals are quasilinear and hence quasiconcave. We therefore have continuous and quasiconcave payoffs in own strategy. Therefore there exists a symmetric pure equilibrium in the game. Equilibria are not unique, as we show in the sequel.

Although much of our analysis is for a finite graph, we can extend the model to allow for infinite graphs as long as payoffs are bounded. Consider the finite truncation of an

[23]See, for example, Becker and Damianov (2006).

infinite invention graph with only T discoveries from the present state (note that the invention graph is a directed acyclic graph by assumption, so this truncation is the tree beginning in the initial state with every branch having length T or less). In this finite truncation, we have already shown equilibria exist. If payoffs are uniformly bounded and there is discounting, then, for T large enough such that the maximal discounted continuation payoff after the T-th discovery is smaller than ε, any equilibria in the finite game with T inventions will also be part of an $\epsilon-$equilibria in the infinite game via the result in Fudenberg and Levine (1986).

7.2 Mixing Equilibria

We say firms are *mixing* when they spread their scientists across multiple projects at a given time. By the usual mixed strategy condition, firms exert effort toward two different inventions only when these two inventions deliver the same payoff.

Let $f_s = w_s + V_{\mathcal{P}s}$. From the proof of Proposition 2, it is easy to see that a firm is indifferent between two states s' and ℓ iff

$$N(\lambda_{s'} f_{s'} - \lambda_\ell f_\ell) = \lambda s' f_{s'} \frac{(M\lambda_{s'} - M\lambda_\ell)}{r + M\lambda_{s'}} \quad \text{(Mix)}.$$

Obviously, when $\lambda_{s'} = \lambda_\ell$ and $f_{s'} = f_\ell$ condition (Mix) holds, because the inventions s' and ℓ are identical in terms of payoffs and simplicities.

Proposition 10. *Suppose inventions s' and ℓ are not identical. Condition (Mix)* **does not** *hold, i.e. there will be no mixing between s' and ℓ if*

1. $(f_{s'} - f_\ell)(\lambda_{s'} - \lambda_\ell) \geq 0,$
2. $(\lambda_{s'} - \lambda_\ell)(\lambda_{s'} f_{s'} - \lambda_\ell f_\ell) < 0,$

Proof. 1. Consider the first part of the proposition.

(a) When $\lambda_{s'} = \lambda_\ell$ condition (Mix) reduces to $f_{s'} = f_\ell$. Therefore, if the inventions are not identical, there will be no mixing between s' and ℓ.

(b) When $f_{s'} = f_\ell$ and $\lambda_{s'} \neq \lambda_\ell$ condition (Mix) reduces to $N = \frac{M\lambda_{s'}}{r+M\lambda_{s'}}$. Since $N > 1 > \frac{M\lambda_{s'}}{r+M\lambda_{s'}}$, this condition does not hold.

(c) Condition (Mix) can be written as

$$N\left(1 - \frac{\lambda_\ell f_\ell}{\lambda_{s'} f_{s'}}\right) = \left(1 - \frac{\lambda_\ell}{\lambda_{s'}}\right) \frac{M\lambda s'}{r + \lambda s'}$$

If $f_{s'} > f_\ell$ and $\lambda_{s'} > \lambda_\ell$, then since $N > 1$, $\frac{M\lambda s'}{r+\lambda s'} < 1$, $\lambda_\ell f_\ell < \lambda_{s'} f_{s'}$ and $\lambda_\ell < \lambda_{s'}$, then condition (Mix) cannot hold. Otherwise,

$$\left(1 - \frac{\lambda_\ell f_\ell}{\lambda_{s'} f_{s'}}\right) < N\left(1 - \frac{\lambda_\ell f_\ell}{\lambda_{s'} f_{s'}}\right) = \left(1 - \frac{\lambda_\ell f_\ell}{\lambda_{s'} f_{s'}}\right) < \left(1 - \frac{\lambda_\ell}{\lambda_{s'}}\right)$$

implying $f_{s'} < f_\ell$, which is a contradiction. Similarly, if $f_{s'} < f_\ell$ and $\lambda_{s'} < \lambda_\ell$ we reach a contradiction.

2. In this case, the lhs of condition (Mix) is non positive and the rhs is strictly positive, and vice-versa.

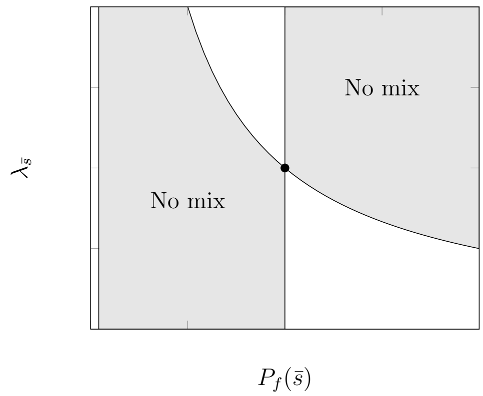
□

This proposition states that firms will never mix between states s' and ℓ if their simplicities are equal but one has higher payoff, or if their payoffs are the same but one is easier to discover than the other, or if one is easier and has higher payoff.

If one invention is easier and a second has a higher payoff inclusive of continuation value, then if firms best respond by mixing between the two, the flow payoff of the easier invention must be strictly higher than the flow payoff of the high payoff invention. In Figure 6, the gray area show inventions $(\lambda_{s'}, P_f(s'))$ that will never mix with the $(\lambda_{\bar{s}}, P_f(\bar{s}))$. This is all to say, large classes of invention graphs have no mixing equilibria.

However, mixing equilibria can exist. It is easiest to see what causes them if we focus on states with no continuation value; in those cases, opponent actions only affect a firm through their cumulative discounted hazard rate, reflected in $\tilde{r} = Nr + N\sum_{z \in S(s)} a_{iz}$. Let $\tilde{r}_{min}$ correspond to all rivals exerting effort towards the hardest invention and $\tilde{r}_{max}$ the corresponding rate when all rivals work on the easiest invention. For any mixture we have $\tilde{r} \in [\tilde{r}_{min}, \tilde{r}_{max}]$.

Figure 6: Regions where simplicities and payoffs where firms will never mix with $(\lambda_{\bar{s}}, P_f(\bar{s}))$



A firm is indifferent between working on inventions k and ℓ iff

$$\frac{\lambda_k \pi_k}{\tilde{r} + \lambda_k} = \frac{\lambda_\ell \pi_\ell}{\tilde{r} + \lambda_\ell} \quad (MC)$$

Therefore if $\frac{\lambda_k \lambda_\ell (\pi_\ell - \pi_k)}{\lambda_k \pi_k - \lambda_\ell \pi_\ell} \in [\tilde{r}_{min}, \tilde{r}_{max}]$ there exists an (inefficient) symmetric mixing equilibrium. For example, if $\lambda_k = 4, \pi_k = 8, \lambda_\ell = 5, , \pi_\ell = 7, r = 1, N = 2$ and $M = 1$, then all firms exerting 1/3 of the effort in k and 2/3 in ℓ is a symmetric mixing equilibrium. By continuity, there is an open set of parameters values with these equilibria.

7.3 Asymmetric Equilibria

We can also construct an asymmetric equilibrium where firms are mixing. Let there be three inventions, and let $\tilde{r}_1$ and $\tilde{r}_2$ be the solutions to

$$\frac{\lambda_k \pi_k}{\tilde{r}_1 + \lambda_k} = \frac{\lambda_\ell \pi_\ell}{\tilde{r}_1 + \lambda_\ell} \quad \text{and} \quad \frac{\lambda_k \pi_k}{\tilde{r}_2 + \lambda_k} = \frac{\lambda_j \pi_j}{\tilde{r}_2 + \lambda_j}.$$

Let firm 1 mix between k and ℓ and firm 2 mix between k and j, accordingly. In this case, we also need to verify that firm 1 does not want to put effort towards j and firm 2 towards ℓ. For example, let $\lambda_k = 6, \pi_k = 3, \lambda_\ell = 12, \pi_\ell = 2, \lambda_j = 2, \pi_j = 6, r = 1, N = 2$ and $M = 1$. Here, firm 1 mixing between k and ℓ exerting 1/2 of the effort in k, and firm 2 mixing between k and j exerting 1/3 of the effort in j is an equilibrium.

7.4 Multiple Equilibria

There further exist small sets of parameters for which there exist multiple equilibria.

Proposition 11. *Consider only two inventions that are perfect substitutes. If $\lambda_k \neq \lambda_\ell$, then there is a region of parameters (π_k, π_ℓ) where there is multiplicity of equilibria with firms allocating effort only towards one invention.*

Proof. Let $M = 1$. All firms putting effort towards ℓ is a symmetric equilibrium if and only if

$$\frac{\lambda_\ell \pi_\ell}{\tilde{r}_\ell + \lambda_\ell} \geq \frac{\lambda_k \pi_k}{\tilde{r}_\ell + \lambda_k}$$

where $\tilde{r}_\ell = rN + (N-1)\lambda_\ell$.

Similarly, all firms putting effort towards k is a symmetric equilibrium iff

$$\frac{\lambda_k \pi_k}{\tilde{r}_k + \lambda_k} \geq \frac{\lambda_\ell \pi_\ell}{\tilde{r}_k + \lambda_\ell}$$

Combining the equations we obtain the inequalities, we obtain that both equilibria exist if and only if

$$\underbrace{\left(\frac{\lambda_\ell}{\lambda_k}\right) \frac{\tilde{r}_k + \lambda_k}{\tilde{r}_k + \lambda_\ell}}_{L_f} \leq \frac{\pi_k}{\pi_\ell} \leq \underbrace{\left(\frac{\lambda_\ell}{\lambda_k}\right) \frac{\tilde{r}_\ell + \lambda_k}{\tilde{r}_\ell + \lambda_\ell}}_{U_f}$$

Notice that $U_f - L_f = \frac{\lambda_\ell}{\lambda_k}(\lambda_k - \lambda_\ell)^2$. Also, we cannot have both $U_f > 1$ and $L_f < 1$.

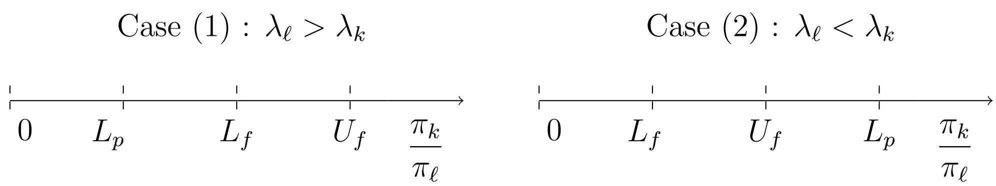
The planner chooses invention k iff

$$\frac{\pi_k}{\pi_\ell} \geq \underbrace{\frac{\lambda_\ell}{\lambda_k}\left(\frac{r + \lambda_k}{r + \lambda_\ell}\right)}_{L_p}.$$

□

If the ratio $\frac{\pi_k}{\pi_\ell}$ is smaller than L_p (L_f), the planner (firm) works on invention ℓ. If the ratio $\frac{\pi_k}{\pi_\ell}$ is larger than L_p (U_f), the planner (firm) works on invention k. There are multiple firm equilibria if the ratio $\frac{\pi_k}{\pi_\ell}$ is in (L_f, U_f). The multiplicity is caused by the following tradeoff. If other firms are all working on the easy project, they are likely to make a discovery quicker than firm i deviating to the hard project. With

Figure 7: Multiplicity of equilibria on perfect substitutes graph



perfect substitutes, if firm i does not discover first, it obtains a payoff of zero from the game. Although deviating can lead to a higher payoff conditional on succeeding first, the probability of being first is smaller. On the other hand, if all rivals are working on the hard project, the potential deviation is to work on an easy project with low payoff, foregoing the higher payoff of the harder project. When the ratios of payoffs and simplicities are structured such that $L_f \leq \frac{\pi_k}{\pi_\ell} \leq U_f$, it is both worth working on the hard project when everyone else does, and worth working on the easy project when everyone else does. As $N \to \infty$ we get $U_f \to L_f$ and the multiplicity disappears.